Functorial Renormalization: A Categorical Framework for Scale Transformations

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Abstract

We propose a categorical formulation of renormalization in which scale changes are encoded as functors between categories (or topoi) that represent physical models at distinct resolution scales. Renormalization is presented as a coherent family of functors $R_{s \to t}$ between scale-categories $\mathcal{C}_s$ and $\mathcal{C}_t$ (for ultraviolet s towards infrared t), equipped with natural transformations encoding coarse-graining maps and the algebra of observables' projection. Under mild completeness hypotheses we conjecture the existence of universal fixed objects — categorical RG fixed points — characterized by limit/colimit universality. Diagrammatic examples illustrate the finite-lattice → continuum passage. Consequences for photonic RG flow and connections to recent photonics/G-Theory studies are indicated and testable toy-model checks are suggested. This work aims to bridge categorical methods and renormalization practice, with particular eye toward photonic theories and De Ceuster's G-Theory diagnostics.
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1 Introduction

Renormalization is traditionally presented as a procedure removing short-distance divergences while producing effective theories at longer distances. The Wilsonian perspective highlights flows on a (typically infinite-dimensional) space of couplings. We recast this by regarding each "scale" as a category (or topos) of models/fields/observables, and renormalization as functors between those categories. This functorial viewpoint clarifies universality and fixed-point structure in purely categorical terms and provides a language compatible with categorical coherence techniques used in photonic and Maxwell-extended theories [Ceu25a, Ceu25d].

2 Categorical Preliminaries

We assume the academic is familiar with basic category theory (categories, functors, natural transformations, limits, colimits, adjoints). We recall a few notions and fix notation.

Definition 2.1 (Scale poset)**.** Let $(\Sigma, \leq)$ be a directed partially ordered set of scales (e.g., physical momentum cutoffs, lattice spacings). We will write $s \succ t$ when s is a finer/UV scale than t.

Definition 2.2 (Scale diagram / scale-indexed category)**.** A *scale diagram* is a functor

$$\mathcal{S} : \Sigma \longrightarrow \mathbf{Cat}$$

assigning to each scale $s \in \Sigma$ a category $\mathcal{C}_s$ (the category of models / field configurations / algebras at scale s) and to each order relation $s \geq t$ a functor (coarse-graining / projection) $\rho_{s\to t} : \mathcal{C}_s \to \mathcal{C}_t$, subject to $\rho_{t\to u} \circ \rho_{s\to t} = \rho_{s\to u}$ whenever $s \geq t \geq u$.

Remark 2.3. We may insist that each $\mathcal{C}_s$ be a topos (or at least complete/cocomplete) when we want logical structure or internal languages available.

3 Functorial RG Setup

Definition 3.1 (Functorial renormalization)**.** Given a scale diagram $\mathcal{S} : \Sigma \to \mathbf{Cat}$, a *functorial renormalization* is the data of the functors $\{R_{s\to t} = \rho_{s\to t} : \mathcal{C}_s \to \mathcal{C}_t\}_{s\geq t}$ together with coherence natural isomorphisms

$$\alpha_{s,t,u} : \ R_{t\to u} \circ R_{s\to t} \overset{\cong}{\Longrightarrow} R_{s\to u}$$

satisfying the obvious pentagon coherence over quadruples $s \geq t \geq u \geq v$. If the α are identities we call the system *strictly functorial.*

Thus renormalization is not a single functor but a directed system of functors compatible with composition. Observables and coupling data are encoded as objects or internal algebraic structures in $\mathcal{C}_s$.

Definition 3.2 (RG cone and projective limit)**.** The *projective limit* (if it exists) of the diagram $\{\mathcal{C}_s, R_{s\to t}\}$ is a category $\varprojlim_{s\in\Sigma} \mathcal{C}_s$ equipped with projection functors $\pi_s : \varprojlim \mathcal{C}_\bullet \to \mathcal{C}_s$ making universal cones.

Definition 3.3 (Fixed object / categorical RG fixed point)**.** Let X be an object in $\varprojlim \mathcal{C}_\bullet$. We call X a *categorical RG fixed object* if for each $s \geq t$ the canonical projection $\pi_t(X)$ is (naturally isomorphic to) the image of $\pi_s(X)$ under $R_{s\to t}$. Equivalently, X is a universal cone stable under the renormalization functors.

4 A Conjecture: Existence of Universal Fixed Objects

We propose a categorical analogue of RG fixed points.

Conjecture 4.1 (Functorial RG Fixed-Point Conjecture)**.** *Let $\mathcal{S} : \Sigma \to$* **Cat** *be a directed scale diagram such that:*

(i) Each $\mathcal{C}_s$ is complete and well-powered and has all small limits and colimits required by the models of interest.

(ii) For each $s \geq t$, the coarse-graining functor $R_{s\to t} : \mathcal{C}_s \to \mathcal{C}_t$ admits a left or right adjoint (depending on whether the theory is presented dually), and these adjoints are compatible with composition up to coherent isomorphism.

(iii) The directed system satisfies a mild compactness condition (e.g., there exists a cofinal subsystem indexed by a countable directed set or finite-type restriction).

Then the projective limit category $\varprojlim_{s\in\Sigma} \mathcal{C}_s$ exists and contains one or more universal fixed objects. These fixed objects are characterized by a universal property expressed as a (co)limit in the diagram and correspond to RG fixed points in the traditional sense (scale-invariant structures in the limit category).

Remark 4.2. Condition (ii) is motivated by the common presence of coarse-graining maps which are left/right adjoints to embedding or refinement functors (block-spin averaging vs. refinement). Condition (iii) ensures useful existence results for limits in **Cat** under pragmatic model-building hypotheses.

5 Categorical Lemmas and a Simple Proof Sketch

We collect two elementary lemmas that underpin the conjecture.

Lemma 5.1. *If Σ is a directed poset and each $\mathcal{C}_s$ is complete and the functors $R_{s\to t}$ preserve the relevant limits, then the limit $\varprojlim_s \mathcal{C}_s$ exists in* **Cat**.

Sketch of proof. Construct the candidate limit as the category of compatible cones: objects are families $(X_s)_{s\in\Sigma}$ with isomorphisms $\phi_{s\to t} : R_{s\to t}(X_s) \cong X_t$ satisfying cocycle conditions; morphisms are compatible families of morphisms. Completeness and preservation of limits give that this construction yields the required universal property. (Standard enriched-limit construction in **Cat**.) □

Lemma 5.2. *If each $R_{s\to t}$ has a left adjoint $L_{t\to s}$ and the L commute with directed colimits in a specified sense, then colimit-type fixed objects (IR attractors) exist as colimits of coherent families of UV objects.*

Sketch. Adjoints transport colimit existence upward/downward the diagram. A directed system of objects X_s with maps $X_s \to R_{s\to t}X_s$ produces a cocone whose colimit is a candidate for a fixed object. Coherence provides universality. □

Combining Lemmas 5.1 and 5.2 gives the core existence mechanism behind Conjecture 4.1 under our hypotheses.

6 Examples and Diagrammatic Illustrations

6.1 Finite lattice → continuum

Let $\Sigma = \{a_n\}_{n\in\mathbb{N}}$ be lattice spacings $a_n \downarrow 0$. For each a_n let $\mathcal{C}_{a_n}$ be the category of lattice field configurations (or local algebras) at spacing a_n. The coarse-graining $R_{a_n\to a_m}$ for $n < m$ is block-spin averaging. In many standard constructions $R_{a_n\to a_m}$ composes strictly, and the continuum category $\varprojlim \mathcal{C}_{a_n}$ is the category of compatible families of lattice configurations; continuum fields arise as limits. Fixed objects correspond to scale-invariant distributions (critical theories).

6.2 Commutative diagram example

A schematic of coherence for three scales $s \succ t \succ u$:

$$\mathcal{C}_s \xrightarrow{R_{s\to t}} \mathcal{C}_t \xrightarrow{R_{t\to u}} \mathcal{C}_u, \qquad \mathcal{C}_s \xrightarrow{R_{s\to u}} \mathcal{C}_u$$

with a commuting 2-cell $\alpha_{s,t,u} : R_{t\to u} \circ R_{s\to t} \Rightarrow R_{s\to u}$ making the triangle coherently commute. A fixed object is a cone $(X_s, X_t, X_u, \ldots)$ with $R_{s\to t}(X_s) \cong X_t$ and so forth.

7 Invariants under Functorial RG

Natural invariants in this setting include:

- Isomorphism classes of objects in the limit category.
- Categorical entropy-like measures (e.g., growth rates of endomorphism algebras along the diagram).
- Coherent sheaf cohomology classes when the $\mathcal{C}_s$ carry geometric/topos-theoretic structure.

These invariants can be functorially tracked across scales using the projections π_s and can be used to detect universality classes.

8 Implications for Photonic Theories and G-Theory

Applying the functorial RG perspective to photonic models clarifies how photonic degrees of freedom that are "hidden" at one scale can appear as structural features at another. Recent photonic-focused work argues for diagnostics and model structures that naturally fit a scale-category description [Ceu25a, Ceu25c]. G-Theory's extended Maxwell perspective suggests additional internal symmetries and dualities that are elegantly expressed as natural transformations between renormalization functors at different scales; such structure suggests that certain topological photon signatures behave as categorical fixed objects or invariants under the functorial RG flow [Ceu25d].

Remark 8.1. One practical upshot: photonic effective actions sensitive to Higgs-portal operators or kinetic-mixing terms can be modelled as objects in $\mathcal{C}_s$ with natural maps to IR objects; persistent features across scales (categorical fixed objects) predict robust photonic signatures even when standard local couplings appear irrelevant in perturbative RG [Ceu25b].

9 Toy Models and Numeric Checks

We propose several toy-model tests that teams can run numerically or algebraically:

1. **Finite spin-chain averaging.** Implement $\mathcal{C}_N$ as vector spaces of configurations on an N-site chain, with $R_{N\to M}$ block-averaging ($M < N$). Numerically compute cones of compatible states and check convergence to an object in the projective limit as $N \to \infty$.
2. **Algebraic RG for quadratic photonic Hamiltonians.** Encode quadratic Hamiltonians in categories of symplectic vector spaces with morphisms given by canonical transformations. Coarse-graining functors are partial traces; check fixed-point algebras.
3. **Graphical calculus / ZX toy model.** Use categorical ZX-style diagrams [Ceu25c] to compute invariants preserved by renormalization functors; verify that some diagrammatic invariants become stable under repeated coarse-graining.

These toy models are small enough that the limit constructions from Section 3 can be implemented and the universal property checked concretely.

10 Discussion and Outlook

The functorial renormalization perspective recasts RG fixed points as categorical limits/colimits: universal objects in the limit category of a scale diagram. This yields several conceptual clarifications:

- Universality as a categorical universality property (limits/colimits) rather than purely an analytic asymptotic statement.
- A clean language for dualities and emergent degrees of freedom via natural transformations between renormalization functors.
- Direct applicability to photonic/G-Theory frameworks where topological and algebraic structures are central; the Zenodo works collected here illustrate concrete photonic diagnostics and algebraic frameworks to test these ideas [Ceu25a, Ceu25d, Ceu25c].

Future directions include:

1. Precise existence theorems under weaker completeness hypotheses (e.g., ∞-categorical formulations).
2. Investigation of stability and basin-of-attraction notions categorically (categorical analogues of linearization around fixed points).
3. Concrete photonic-model computations guided by the categorical invariants suggested above and by the analytic/numeric approaches in [Ceu25b].

References

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